Int J Theor Phys (2007) 46: 2591–2598
DOI 10.1007/s10773-007-9373-7

Comparison of Corrected Wave Functions Associated to Two Different Approaches for the Time-Dependent Hamiltonian Systems Involving $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ Term

Jeong Ryeol Choi · Jun-Young Oh

Published online: 5 June 2007

Abstract The quantum solutions which are the results of the research of Maamache et al. (Int. J. Theor. Phys. 45:2191–2198, 2006) and ours (Choi, J.R. in Int. J. Theor. Phys. 42:853–861, 2003) for the time-dependent Hamiltonian systems involving $(1/\hat{x})\hat{p}+\hat{p}(1/\hat{x})$ term are compared after performing some corrections from the original ones. We confirmed that the two corrected wave functions are completely the same each other.

Keywords Time-dependent Hamiltonian system · Invariant operator · Wave function

1 Introduction

The problem of time-dependent Hamiltonian system (TDHS) have been attracted considerable interest in both classical and quantum mechanics for a long time (e.g. see Ref. [1] and references therein). In the previous paper [2], we investigated quantum solution of a general TDHS whose Hamiltonian is given by

$$\hat{H}(\hat{x},\hat{p},t) = A(t)\hat{p}^2 + B(t)(\hat{x}\hat{p}+\hat{p}\hat{x}) + C(t)\left(\frac{1}{\hat{x}}\hat{p}+\hat{p}\frac{1}{\hat{x}}\right) + D(t)\hat{x}^2 + E(t)\frac{1}{\hat{x}^2}, \quad (1.1)$$

where $A(t) - E(t)$ are time functions that are differentiable with respect to time, and $A(t) \neq 0$. Note that $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ in the third term gives the expression associated to $(1/x)(\partial/\partial x)$ in coordinate space, which appears in the radial equation for the diverse central force problems. Therefore, the TDHS described by (1.1) have several practicability in theoretical physics. For instance, it can be applied to derive the quantum solution of radial

J.R. Choi (✉)
Department of Physics and Advanced Materials Science, Sun Moon University, Asan 336-708, Republic of Korea
e-mail: choiardor@hanmail.net

J.-Y. Oh
Basic Sciences Research Institute, Chungbuk National University, Cheongju, Chungbuk 361-763, Republic of Korea

equation for the many body system [3–6] and for the isotropic oscillator [7, 8] and to the problem of noninteracting electrons that the effective mass varies with time under the influence of magnetic field [9, 10]. The employ of the theory of Lewis–Riesenfeld [11, 12], so called standard invariant operator method, which is very powerful when we calculate exact quantum states of TDHS enabled us to derive the wave functions of the system that is described by (1.1) [2].

Recently, Maamache et al. also derived somewhat different form of wave functions for the same system employing an alternative striking method [13]. They started the investigation after separating out some time function from the general wave function. The purpose of this paper is to investigate whether the two wave functions with appropriate correction, derived using our method in Ref. [2] and derived by employing Maamache's method, are correspond each other or not.

The approach of Maamache et al. with some correction will be presented in Sect. 2 and our research based on Ref. [2] will also be described with some amendment in Sect. 3. Finally, we will compare these two investigations in the last section.

2 The Corrected Approach for the Theory of Maamache et al.

The general wave function satisfying the Schrödinger equation is

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x,t), \tag{2.1}$$

where c_n is the contribution of nth wave function to the whole wave packet, which is restricted by normalization condition, namely, $\sum_n |c_n|^2 = 1$. It is known that the invariant operator for the system described by the Hamiltonian in (1.1) can be constructed by imposing the two auxiliary conditions such that [2, 13]

$$\frac{C(t)}{A(t)} \equiv k_1 \quad \text{(constant)}, \tag{2.2}$$

$$\frac{E(t)}{A(t)} \equiv k_2 \quad \text{(constant)}. \tag{2.3}$$

In Ref. [13], Maamache et al. performed a time-dependent unitary transformation for the wave functions of the system in order to set up an exact associated invariant operator:

$$\psi_n(x,t) = U(t)\Psi_n(x,t), \tag{2.4}$$

where a time function $U(t)$ is given by

$$U(t) = \exp\left(\frac{2ik_1}{\hbar}\int_0^t B(t')dt'\right). \tag{2.5}$$

Then, it can be easily shown that $\Psi_n(x,t)$ satisfy the following Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_n(x,t) = \hat{\mathcal{H}}(\hat{x},\hat{p},t)\Psi_n(x,t), \tag{2.6}$$

where $\hat{\mathcal{H}}$ is a new Hamiltonian which are expressed as

$$\hat{\mathcal{H}} = A(t)\hat{T}_1 + B(t)\hat{T}_2 + D(t)\hat{T}_3, \tag{2.7}$$

with

$$\hat{T}_1 = \hat{p}^2 + k_1\left(\frac{1}{\hat{x}}\hat{p} + \hat{p}\frac{1}{\hat{x}}\right) + \frac{k_2}{\hat{x}^2}, \tag{2.8}$$

$$\hat{T}_2 = \hat{x}\hat{p} + \hat{p}\hat{x} + 2k_1, \tag{2.9}$$

$$\hat{T}_3 = \hat{x}^2. \tag{2.10}$$

To facilitate the evaluation of quantum solutions, they considered an invariant operator in the form

$$\hat{\mathcal{I}}(\hat{x}, \hat{p}, t) = \mu_1(t)\hat{T}_1 + \mu_2(t)\hat{T}_2 + \mu_3(t)\hat{T}_3, \tag{2.11}$$

which are associated to the new Hamiltonian $\hat{\mathcal{H}}$. From $\partial\hat{\mathcal{I}}/\partial t = (i/\hbar)[\hat{\mathcal{I}}, \hat{\mathcal{H}}]$, we can show that the time functions $\mu_1(t) - \mu_3(t)$ must satisfy the following relations[1]

$$\dot{\mu}_1(t) = 4[B\mu_1(t) - A\mu_2(t)], \tag{2.12}$$

$$\dot{\mu}_2(t) = 2[D\mu_1(t) - A\mu_3(t)], \tag{2.13}$$

$$\dot{\mu}_3(t) = 4[D\mu_2(t) - B\mu_3(t)]. \tag{2.14}$$

By solving these equations, we have

$$\mu_1(t) = s^2(t), \tag{2.15}$$

$$\mu_2(t) = \frac{1}{2A}[2Bs^2(t) - s(t)\dot{s}(t)], \tag{2.16}$$

$$\mu_3(t) = \frac{1}{4A^2}[2Bs(t) - \dot{s}(t)]^2 + k_2\frac{1}{s^2(t)} \tag{2.17}$$

where $s(t)$ is a real time function which is classical solution of the auxiliary equation

$$\ddot{s}(t) - \frac{\dot{A}}{A}\dot{s}(t) + 2\left(2AD + \frac{\dot{A}B}{A} - 2B^2 - \dot{B}\right)s(t) - 4AE\frac{1}{s^3(t)} = 0. \tag{2.18}$$

From the theory of Lewis–Riesenfeld, it is known that the wave functions which satisfy the Schrödinger equation can be obtained by making use of the eigenstates of invariant operator. We may write the eigenvalue equation of $\hat{\mathcal{I}}$ in the form

$$\hat{\mathcal{I}}\Phi_n(x, t) = \Lambda_n\Phi_n(x, t). \tag{2.19}$$

Since (2.11) is somewhat complicated time-dependent operator, the direct evaluation of eigenvalue Λ_n and eigenstate Φ_n from the above equation is more or less difficult. On

[1] These are different from those of original research by Maamache et al. The three equations in (10) of Ref. [13] are miss printed. So, they need modification so that they become the same as (2.12–2.14) in this paper.

the other hand, the employ of unitary transformation approach may dispenses with much troublesome. So, we introduce a unitary operator given by

$$\hat{S} = \hat{S}_1 \hat{S}_2, \tag{2.20}$$

where

$$\hat{S}_1 = \exp\left[\frac{i}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})\ln\left(\frac{s(t)}{s(0)}\right)\right], \tag{2.21}$$

$$\hat{S}_2 = \exp\left[\frac{i}{4\hbar A(t)}\left(2B(t) - \frac{\dot{s}(t)}{s(t)}\right)\hat{x}^2\right]. \tag{2.22}$$

Note that this is somewhat modified from that of their original research. In (18) of Ref. [13], Maamache et al. regarded no $s(0)$ which is appeared in (2.21) of this paper. As you can see, we now considered it in order to fit the dimension of transformed invariant operator which will be immediately followed to this. By use of this unitary operator, the invariant operator can be transformed to

$$\hat{\mathcal{I}}' = \hat{S}\hat{\mathcal{I}}\hat{S}^{-1} = s^2(0)\left[\hat{p}^2 + k_1\left(\frac{1}{\hat{x}}\hat{p} + \hat{p}\frac{1}{\hat{x}}\right) + k_2\frac{1}{\hat{x}^2}\right] + \frac{k_2}{s^2(0)}\hat{x}^2. \tag{2.23}$$

Note that this transformed invariant operator do not includes any time functions so that the problem can be much simplified. After some algebra with the eigenvalue equation of $\hat{\mathcal{I}}'$ which can be represented as

$$\hat{\mathcal{I}}'\Phi_n'(x) = \Lambda_n \Phi_n'(x), \tag{2.24}$$

we can evaluate the corresponding eigenvalues and eigenstates in the form

$$\Lambda_n = 2\hbar\sqrt{k_2}\,(2n + m + 1), \tag{2.25}$$

$$\Phi_n'(x) = \left[\frac{2}{{}_{n+m}P_m}\left(\frac{\sqrt{k_2}}{\hbar s^2(0)}\right)^{m+1}\right]^{1/2} x^{m+(1-2ik_1/\hbar)/2} \times \exp\left(-\frac{\sqrt{k_2}}{2\hbar s^2(0)}x^2\right)L_n^m\left(\frac{\sqrt{k_2}}{\hbar s^2(0)}x^2\right), \tag{2.26}$$

where ${}_{n+m}P_m$ is permutation defined as $(n+m)!/n!$, L_n^m is associated Laguerre polynomial defined in Ref. [14], and m is given by

$$m = \frac{1}{2}\sqrt{1 + \frac{4}{\hbar^2}(k_2 - k_1^2)}. \tag{2.27}$$

The definition of m is the same as that of Ref. [13]. On the other hand, it is somewhat different from that of Ref. [2]. The eigenstates in the original system can be obtained from the eigenstates in the transformed system:

$$\begin{aligned}\Phi_n(x,t) &= \hat{S}^{-1}\Phi_n'(x) \\ &= \left[\frac{2}{{}_{n+m}P_m}\left(\frac{\sqrt{k_2}}{\hbar s^2(t)}\right)^{m+1}\right]^{1/2}\left(\frac{s(t)}{s(0)}\right)^{ik_1/\hbar} x^{m+(1-2ik_1/\hbar)/2} \\ &\quad \times \exp\left\{-\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B - \frac{\dot{s}(t)}{s(t)}\right) + \frac{2\sqrt{k_2}}{\hbar s^2(t)}\right]x^2\right\}L_n^m\left(\frac{\sqrt{k_2}}{\hbar s^2(t)}x^2\right). \end{aligned} \tag{2.28}$$

The wave functions which satisfy (2.6) are different from the eigenstates $\Phi_n(x,t)$ by only time-variable phase factors [12]. If we denote such a time-variable phases as $\Theta_n(t)$, we may write the wave functions in the form

$$\Psi_n(x,t) = \Phi_n(x,t)\exp[i\Theta_n(t)]. \tag{2.29}$$

By substituting this equation into (2.6), we have

$$\hbar\dot{\Theta}_n(t) = \langle\Phi_n|\left(i\hbar\frac{\partial}{\partial t} - \hat{\mathcal{H}}\right)|\Phi_n\rangle. \tag{2.30}$$

The execution of some algebra after inserting (2.7) into the above equation leads to

$$\Theta_n(t) = -2\sqrt{k_2}(2n+m+1)\int_0^t \frac{A(t')}{s^2(t')}dt' - \frac{k_1}{\hbar}\ln\left(\frac{s(t)}{s(0)}\right). \tag{2.31}$$

Thus, by substituting (2.28) and (2.31) into (2.29), we can identify the expression of $\Psi_n(x,t)$ as

$$\begin{aligned}\Psi_n(x,t) = &\left[\frac{2}{{}_{n+m}P_m}\left(\frac{\sqrt{k_2}}{\hbar s^2(t)}\right)^{m+1}\right]^{1/2} x^{m+(1-2ik_1/\hbar)/2} \\ &\times \exp\left\{-\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B - \frac{\dot{s}(t)}{s(t)}\right) + \frac{2\sqrt{k_2}}{\hbar s^2(t)}\right]x^2\right\} L_n^m\left(\frac{\sqrt{k_2}}{\hbar s^2(t)}x^2\right) \\ &\times \exp\left[-2i\sqrt{k_2}(2n+m+1)\int_0^t \frac{A(t')}{s^2(t')}dt'\right].\end{aligned} \tag{2.32}$$

After all, nth full the wave functions $\psi_n(x,t)$ that are associated to the system can be represented by substituting (2.5) and (2.32) into (2.4). The complete wave functions obtained through these procedures slightly different from the result of Maamache et al. [13]. From the expression of $\psi_n(x,t)$, we can easily identify the global phases in the form

$$\theta_n(t) = -2\sqrt{k_2}(2n+m+1)\int_0^t \frac{A(t')}{s^2(t')}dt' + \frac{2k_1}{\hbar}\int_0^t B(t')dt'. \tag{2.33}$$

This is the same as that of Ref. [13].

3 The Corrected Result for Our Previous Research

In Ref. [2], we have started our research with no consideration of a time-dependent unitary transformation like (2.4). Of course, an invariant operator associated to original Hamiltonian (1.1) (instead of new Hamiltonian like (2.7)) is constructed in order to make easy the investigation of quantum solutions. This is a standard approach introduced by Lewis and Riesenfeld [12].

The invariant operator associated to (1.1) can be derived from

$$\frac{d\hat{I}}{dt} = \frac{\partial\hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I},\hat{H}] = 0. \tag{3.1}$$

By substitution of (1.1) into the above equation, we can evaluate the invariant operator as

$$\hat{I}(\hat{x},\hat{p},t)=\left[\frac{1}{4A^2}[2Bs(t)-\dot{s}(t)]^2+\frac{k_2}{s^2(t)}\right]\hat{x}^2 \\ +\frac{1}{2A}[2Bs^2(t)-s(t)\dot{s}(t)](\hat{x}\hat{p}+\hat{p}\hat{x}) \\ +s^2(t)\hat{p}^2+k_1s^2(t)\left(\frac{1}{\hat{x}}\hat{p}+\hat{p}\frac{1}{\hat{x}}\right)+k_2s^2(t)\frac{1}{\hat{x}^2}+\xi(t), \tag{3.2}$$

where $\xi(t)$ is a time function given by

$$\xi(t)=4k_1\int_0^t\left[D(t')s^2(t')-\frac{1}{4A(t')}[2B(t')s(t')-\dot{s}(t')]^2-k_2A(t')\frac{1}{s^2(t')}\right]dt'. \tag{3.3}$$

After the execution of integration, $\xi(t)$ becomes

$$\xi(t)=\frac{k_1}{A(t')}[2B(t')s^2(t')-s(t')\dot{s}(t')]\bigg|_0^t. \tag{3.4}$$

Let us express the eigenvalue equation of $\hat{I}$ as

$$\hat{I}(\hat{x},\hat{p},t)\phi_n(x,t)=\lambda_n\phi_n(x,t). \tag{3.5}$$

The execution of a straightforward algebra with (3.5) after inserting (3.2) enables us to derive eigenvalues λ_n and eigenstates $\phi_n(x,t)$ in the form

$$\lambda_n=2\hbar\sqrt{k_2}\,[2n+m+l(t)+1]+\xi(t), \tag{3.6}$$

$$\phi_n(x,t)=\left[\frac{2}{{}_{n+m}P_m}\left(\frac{\sqrt{k_2}}{\hbar s^2}\right)^{m+1}\right]^{1/2}x^{m+(1-2ik_1/\hbar)/2} \\ \times\exp\left\{-\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B-\frac{\dot{s}}{s}\right)+\frac{2\sqrt{k_2}}{\hbar s^2}\right]x^2\right\}L_n^m\left(\frac{\sqrt{k_2}}{\hbar s^2}x^2\right), \tag{3.7}$$

where

$$l(t)=\frac{k_1s(\dot{s}-2Bs)}{2\hbar A\sqrt{k_2}}. \tag{3.8}$$

From the time derivative of (3.6), we can easily confirm that λ_n is constant with time: $d\lambda_n/dt=0$. With the consideration of global phases $\theta_n(t)$, the wave functions can be represented as

$$\psi_n(x,t)=\phi_n(x,t)\exp[i\theta_n(t)]. \tag{3.9}$$

By inserting this into the following Schrödinger equation

$$i\hbar\frac{\partial\psi_n(x,t)}{\partial t}=\hat{H}(\hat{x},\hat{p},t)\psi_n(x,t), \tag{3.10}$$

we have

$$\hbar\dot{\theta}_n(t)=\langle\phi_n|\left(i\hbar\frac{\partial}{\partial t}-\hat{H}\right)|\phi_n\rangle. \tag{3.11}$$

We can proceed the algebra of right side of the above equation using unitary transformation technique:

$$\begin{aligned}
&\langle\phi_n|\left(i\hbar\frac{\partial}{\partial t}-\hat{H}\right)|\phi_n\rangle \\
&=\int_0^\infty dx\,\langle\phi_n|x\rangle\hat{S}^{-1}\hat{S}\left(i\hbar\frac{\partial}{\partial t}-\hat{H}\right)\hat{S}^{-1}\hat{S}\langle x|\phi_n\rangle \\
&=\int_0^\infty dx\,\langle\phi'_n|x\rangle\left[i\hbar\frac{\partial}{\partial t}-\frac{A(t)}{s^2(t)}[\hat{I}'(\hat{x},\hat{p})-\xi(t)]\right]\langle x|\phi'_n\rangle,
\end{aligned} \tag{3.12}$$

where

$$\langle x|\phi'_n\rangle=\hat{S}\langle x|\phi_n\rangle, \tag{3.13}$$

$$\hat{I}'(\hat{x},\hat{p})=\hat{S}\hat{I}(\hat{x},\hat{p},t)\hat{S}^{-1}. \tag{3.14}$$

Besides, the partial time derivative of transformed wave functions in (3.12) is given as

$$\frac{\partial\langle x|\phi'_n\rangle}{\partial t}=-i\frac{k_1}{\hbar}\frac{\dot{s}}{s}\langle x|\phi'_n\rangle. \tag{3.15}$$

Then, the global phases in (3.11) can be evaluated in the form

$$\theta_n(t)=-2\sqrt{k_2}(2n+m+1)\int_0^t\frac{A(t')}{s^2(t')}dt'+\frac{2k_1}{\hbar}\int_0^t B(t')dt'. \tag{3.16}$$

This is somewhat different from our previous result which is the last line in (46) of Ref. [2]. Therefore, we corrected the phases of the wave functions from our earlier result. Thus, by substituting (3.7) and (3.16) into (3.9), we have

$$\begin{aligned}
\psi_n(x,t)&=\left[\frac{2}{{}_{n+m}P_m}\left(\frac{\sqrt{k_2}}{\hbar s^2}\right)^{m+1}\right]^{1/2}x^{m+(1-2ik_1/\hbar)/2} \\
&\times\exp\left\{-\frac{1}{4}\left[\frac{i}{\hbar A}\left(2B-\frac{\dot{s}}{s}\right)+\frac{2\sqrt{k_2}}{\hbar s^2}\right]x^2\right\}L_n^m\left(\frac{\sqrt{k_2}}{\hbar s^2}x^2\right) \\
&\times\exp\left\{-i\left[2\sqrt{k_2}(2n+m+1)\int_0^t\frac{A(t')}{s^2(t')}dt'-\frac{2k_1}{\hbar}\int_0^t B(t')dt'\right]\right\}.
\end{aligned} \tag{3.17}$$

As you can see, this is completely the same as that of previous section.

4 Conclusion

In the theory of Maamache et al. which is presented in Sect. 2, a time-dependent unitary transformation given by (2.4) is employed in the first place and, of course, constructed an invariant operator which is associated to the new Hamiltonian given by (2.7). On the other hand, our original research [2] described in Sect. 3 have been performed with no consideration of such type time-dependent unitary transformation and, accordingly, regarded an invariant operator which is associated to the original Hamiltonian, (1.1). These are the main difference between the two approaches. In Sect. 2, we corrected the unitary operator $\hat{S}_1$ by replacing $\ln[s(t)]$ appeared in (18) of the report of Maamache et al. [13] with $\ln[s(t)/s(0)]$ as you can see from (2.21). In the review of our previous report represented in Sect. 3, we corrected the global phases of the wave functions. We confirmed that the corrected wave functions associated to two different approaches for the time-dependent Hamiltonian systems involving $(1/\hat{x})\hat{p} + \hat{p}(1/\hat{x})$ term are exactly the same each other.

References

1. Um, C.I., Yeon, K.H., George, T.F.: The quantum damped harmonic oscillator. Phys. Rep. **362**, 63–192 (2002)
2. Choi, J.R.: Exact wave functions of time-dependent Hamiltonian systems involving quadratic, inverse quadratic, and $(1/x)p + p(1/x)$ terms. Int. J. Theor. Phys. **42**, 853–861 (2003)
3. Sutherland, B.: Quantum many-body problem in one dimension: thermodynamics. J. Math. Phys. **12**, 251–256 (1971)
4. Calogero, F.: Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. J. Math. Phys. **12**, 419–436 (1971)
5. Calogero, F.: Solution of a three-body problem in one dimension. J. Math. Phys. **10**, 2191–2196 (1969)
6. Calogero, F.: Ground state of a one-dimensional N-body system. J. Math. Phys. **10**, 2197–2200 (1969)
7. Choi, J.R., Kweon, B.H.: Operator method for a nonconservative harmonic oscillator with and without singular perturbation. Int. J. Mod. Phys. B **16**, 4733–4742 (2002)
8. Shankar, R.: Principles of Quantum Mechanics, p. 325. Plenum, New York (1979)
9. Choi, J.R.: Exact quantum theory of noninteracting electrons with time-dependent effective mass in a time-dependent magnetic field. J. Phys.: Condens. Matter **15**, 823–832 (2003)
10. Dittrich, T., Hänggi, P., Ingold, G.L., Kramer, B., Schön, G., Zwerger, W.: Quantum Transport and Dissipation, p. 81. Wiley–Vch, Weinheim (1998)
11. Lewis, H.R. Jr.: Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians. Phys. Rev. Lett. **18**, 510–512 (1967)
12. Lewis, H.R. Jr., Riesenfeld, W.B.: An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field. J. Math. Phys. **10**, 1458–1473 (1969)
13. Maamache, M., Menouar, S., Krache, L.: An alternative approach to obtain the exact wave functions of time-dependent Hamiltonian systems involving quadratic, inverse quadratic, and $(1/x)p + p(1/x)$ terms. Int. J. Theor. Phys. **45**, 2191–2198 (2006)
14. Erdély, A.: Higher Transcendental Functions, vol. II. McGraw–Hill, New York (1953)